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# Vertical dispersion by stratified turbulence

E. LINDBORG AND G. BRETHOUWER

Linné Flow Centre, Department of Mechanics, KTH, SE-100 44 Stockholm, Sweden

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We derive a relation for the growth of the mean square of vertical displacements, $\delta z$, of fluid particles of stratified turbulence. In the case of freely decaying turbulence, we find that for large times $\langle \delta z^2 \rangle$ goes to a constant value $2(E_P(0) + aE(0))/N^2$, where $E_P(0)$ and $E(0)$ are the initial mean potential and total turbulent energy per unit mass, respectively, $a < 1$ and $N$ is the Brunt–Väisälä frequency. In the case of stationary turbulence, we find that $\langle \delta z^2 \rangle = \langle \delta b^2 \rangle / N^2 + 2\epsilon_P t/N^2$, where $\epsilon_P$ is the mean dissipation of turbulent potential energy per unit mass and $\langle \delta b^2 \rangle$ is the Lagrangian structure function of normalized buoyancy fluctuations. The first term is the same as that obtained in the case of adiabatic fluid particle dispersion. This term goes to the finite limit $4E_P/N^2$ as $t \to \infty$. Assuming that the second term represents irreversible mixing, we show that the Osborn & Cox model for vertical diffusion is retained. In the case where the motion is dominated by a turbulent cascade with an eddy turnover time $T \gg N^{-1}$, rather than linear gravity waves, we suggest that there is a range of time scales, $t$, between $N^{-1}$ and $T$, where $\langle \delta b^2 \rangle = 2\pi C_{PL} \epsilon_P t$, where $C_{PL}$ is a constant of the order of unity. This means that for such motion the ratio between the adiabatic and the diabatic mean-square displacement is universal and equal to $\pi C_{PL}$ in this range. Comparing this result with observations, we make the estimate $C_{PL} \approx 3$.

## 1. Introduction

The atmosphere and the oceans are mixed by random vertical velocity fluctuations. This mixing is commonly modelled by introducing the concept of eddy diffusion. Small-scale eddies are thought to cause large-scale effects in the same statistical way as molecular motions at a microscopic level can give rise to macroscopic fluctuations. For stable stratification one of the most commonly used eddy diffusion models is the Osborn & Cox (1972) model, with eddy diffusivity

$$D_{\mathscr{E}} = \frac{\epsilon_P}{N^2} = c\frac{\epsilon_K}{N^2}, \tag{1.1}$$

where $N$ is the Brunt–Väisälä frequency, $\epsilon_P$ is the mean potential energy dissipation per unity mass, $\epsilon_K$ is the mean kinetic energy dissipation and $c = \epsilon_P/\epsilon_K$ is a number which is commonly referred to as the ‘mixing efficiency’ (Osborn 1980). The Osborn & Cox eddy diffusivity is estimated from measurements of $\epsilon_K$ and $N$ and then assigning a value to $c$. Much experimental and numerical effort has been made to determine the value of $c$ (see e.g. Peltier & Caulfield 2003 and Pardyjak, Monty & Fernando 2002). The current standard value is $c = 0.2$.

The Osborn & Cox model is derived by assuming that there is a balance between dissipation and production in the Reynolds-averaged equation for mean-square buoyancy fluctuations. For many geophysical flows it can be questioned whether there is such a balance. A great deal of effort (e.g. Lien & D’Asaro 2004) has therefore been made to analyse the problem of vertical mixing from the fundamental point of view of fluid particle dispersion. The microscopic basis of molecular diffusion is the random walk of molecules with a mean-square displacement growing linearly with time. The diffusivity can be determined from the rate of this growth (Einstein 1905). If it could be demonstrated that the mean square of vertical displacements of fluid particles, $\delta z$, in a geophysical flow grows linearly with time for sufficiently small time scales, which may still be larger than $N^{-1}$, then it would seem natural to define the eddy diffusivity as

$$\lim_{\substack{t\to 0\\ t>N^{-1}}} \frac{1}{2}\frac{\langle \delta z^2\rangle}{t}, \tag{1.2}$$

in close analogy with the corresponding relation for the molecular diffusivity (Einstein 1905; see also Kennard 1938, p. 286). However, there is an obvious problem with such an approach. It cannot generally be assumed that the vertical fluid particle movements are completely random, like molecular movements. A fluid particle which is adiabatically displaced in a stably stratified fluid will experience a restoring force. For short time scales the movement of the particle may be similar to a random walk, but eventually the particle will always return to the same level that it started from. Therefore, adiabatic motion will not produce any net gradient diffusion. The problem has been analysed by Winters & D'Asaro (1996) and Lien & D'Asaro (2004), who developed analytical and experimental methods by which one may separate the adiabatic part of the motion from the diabatic part. In this paper, we will show that that the expression for the growth of mean-square particle displacements consists of two terms, which can be interpreted as an adiabatic and a diabatic contribution. Indeed, the diabatic contribution is found to grow linearly with time and substituting this linear growth into the formula (1.2) we obtain the Osborn & Cox (1972) model. We will also make the suggestion that for time scales between $N^{-1}$ and $T$, where $T$ is of the order of one day in geophysical applications, the adiabatic mean-square displacement growth is also linear, and that the ratio between the adiabatic and diabatic growth is universal in this range.

## 2. Analysis

### 2.1. *Fluid particle dispersion*

In the case of a constant background stratification the Boussinesq equations can be written as

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\nabla p - N\boldsymbol{e}_z b + \nu\nabla^2\boldsymbol{u}, \tag{2.1}$$

$$\frac{\mathrm{d}b}{\mathrm{d}t} = Nw + \kappa\nabla^2 b, \tag{2.2}$$

$$\nabla\cdot\boldsymbol{u} = 0. \tag{2.3}$$

Here, $\boldsymbol{u}$ is the velocity field, $\nu$ and $\kappa$ are the molecular viscosity and diffusivity, respectively, $p$ is the pressure, $\boldsymbol{e}_z$ is the vertical unit vector, $N$ is the Brunt–Väisälä frequency, $b=-g\rho'/(N\rho_o)$, where $\rho'$ and $\rho_o$ are the fluctuating and background densities, respectively, $g$ is the acceleration due to gravity and $w=\boldsymbol{e}_z\cdot\boldsymbol{u}$ is the vertical velocity component. For applications in the atmosphere, the fluctuating density should be replaced by the fluctuating potential temperature, including a minus sign, and the background density field should be replaced by the background potential temperature field. The normalization of the buoyancy so that it has the dimension of velocity, rather than acceleration, as is common practice, allows us to write mean potential energy per unit mass and mean dissipation of potential energy per unit mass as

$$E_P = \tfrac{1}{2}\langle b^2\rangle, \tag{2.4}$$

$$\epsilon_P = \kappa\langle \nabla b \boldsymbol{\cdot} \nabla b\rangle. \tag{2.5}$$

The assumption of a constant Brunt–Väisälä frequency does not imply that it should be regarded as globally constant over very long time and length scales, but only that it can be regarded as approximately constant over the time and length scales which will be considered in this analysis. Three time scales will appear in the analysis: the Kolmogorov time scale $\tau = (\nu/\epsilon_K)^{1/2}$, the buoyancy time scale $N^{-1}$ and the eddy turnover time scale $T$, which in geophysical applications should scale with the inertial frequency $f_0$ (Lindborg 2005) and therefore should be of the order of one day. We will assume that $\tau \ll N^{-1} \ll T$. In the case where we take the limit $t/T \to \infty$, the assumption of a constant $N$ must be regarded as an idealization.

We consider a domain in a fluid which is governed by equations (2.1)–(2.3) and we assume that there are no rigid boundaries in the vicinity of the domain. The width of the domain is larger than $l_h = E_K^{3/2}/\epsilon_K$ in both horizontal directions and the height is larger than $l_v = E_K^{1/2}/N$, where $E_K$ is the mean turbulent kinetic energy per unit mass. Estimates of these length scales for geophysical applications will be given in §2.2. Following Pearson, Puttock & Hunt (1983), we integrate (2.2) along the track of a fluid particle from time zero to $t$, multiply the resulting equation by $w(t)$ and average over all fluid particles in the domain. In this way we obtain

$$\langle b(t)w(t)\rangle - \langle b(0)w(t)\rangle = N\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \delta z^2\rangle + \kappa\int_0^t \langle \nabla^2 b(t')w(t)\rangle\,\mathrm{d}t', \tag{2.6}$$

where $\delta z = z(t) - z(0)$. Again using (2.2) we can rewrite the left-hand side of equation (2.6) as

$$\frac{1}{N}\left[\frac{\mathrm{d}E_P}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\langle b(0)b(t)\rangle + \epsilon_P + \kappa\langle b(0)\nabla^2 b(t)\rangle\right]. \tag{2.7}$$

Here, we have assumed that the small-scale turbulent field is locally homogeneous (Kolmogorov 1941), which implies that

$$-\kappa\langle b\nabla^2 b\rangle = \kappa\langle \nabla b \boldsymbol{\cdot} \nabla b\rangle = \epsilon_P. \tag{2.8}$$

The last term within the square brackets in (2.7) can be written as

$$\kappa\langle b(0)\nabla^2 b(t)\rangle = -\epsilon_P F_1(t/\tau), \tag{2.9}$$

where $F_1(0) = 1$ by local homogeneity. The presence of the Laplace operator in the two-time correlation (2.9) suggests that it is a correlation which is determined by the dynamics of the very smallest turbulent scales. We therefore assume that the correlation time is equal to the Kolmogorov time scale so that $|F_1| \ll 1$ when $t \gg \tau$. This means that the last term in (2.7) is much smaller than the sum of the three other terms when $t \gg \tau$, unless these three terms cancel each other. In the Appendix, we argue that there is no such cancellation.

We now turn to the last term on the right-hand side of equation (2.6). Pearson *et al.* (1983) argued that in the case of stable stratification this term should be of leading order. We shall argue that it is negligible for $t \gg \tau$. First we make an order of magnitude estimate of the leading-order contribution to this term and argue that this is small. Then we argue that the leading-order contribution, in fact, vanishes if we make the assumption of local isotropy (Kolmogorov 1941). The two-time Lagrangian correlation function $\kappa\langle\nabla^2 b(t')w(t)\rangle$ has the same dimension as energy dissipation per unit mass and should scale with the mean dissipation. The presence of the Laplace operator in this term suggests that the correlation time is very small, presumably equal to the Kolmogorov time scale. We can therefore write

$$\kappa\langle\nabla^2 b(t')w(t)\rangle = \epsilon_K F_2\left(\frac{t-t'}{\tau}\right), \tag{2.10}$$

where we also have assumed that the turbulence is a stationary process at the Kolmogorov scale. Using expression (2.10) we find

$$\kappa\int_0^t \langle\nabla^2 b(t')w(t)\rangle\,\mathrm{d}t' = (\epsilon_K\nu)^{1/2}\int_0^{t/\tau} F_2(\lambda)\,\mathrm{d}\lambda \sim (\epsilon_K\nu)^{1/2}, \tag{2.11}$$

when $t \gg \tau$. Here we have assumed that the two-time correlation (2.10) goes to zero sufficiently fast for large time separations, so that $F_2$ is integrable.

The left-hand side of (2.6) is of the order of $\epsilon_P/N \sim \epsilon_K/N$. The ratio between (2.11) and the left-hand side of (2.6) can thus be estimated as $(\epsilon_K/N^2\nu)^{-1/2} = \tau N \ll 1$. The last term on the right-hand side of (2.6) is therefore negligible. As already pointed out, making the assumption of local isotropy, it can be argued that this term is not only small but zero. This can be seen from the following estimate:

$$|\langle\nabla^2 b(t')w(t)\rangle| \lesssim |\langle\nabla^2 b(t)w(t)\rangle| = |\langle\nabla b(t)\cdot\nabla w(t)\rangle| = 0, \tag{2.12}$$

where the two last equalities follow from local isotropy. Local isotropy implies that any two-point correlation of derivatives of flow field quantities is isotropic, meaning that it is invariant under arbitrary rotations and reflections (see Monin & Yaglom 1975). In stratified turbulence the local isotropy assumption is valid at length scales which are considerably smaller than the Ozmidov length scale $l_O = \epsilon_K^{1/2}/N^{3/2}$ (see Brethouwer *et al.* 2007). The last equality in (2.12) follows by imposing invariance under reflection in a horizontal plane, that is invariance under the transformation $(z, w) \rightarrow (-z, -w)$.

Substituting (2.7) into (2.6), neglecting the two terms including $\kappa$ and integrating we obtain

$$\langle\delta z^2\rangle = \frac{2}{N^2}\left(E_P(0) + E_P(t) - \langle b(0)b(t)\rangle + \int_0^t \epsilon_P(t')\,\mathrm{d}t'\right). \tag{2.13}$$

Before analysing the case of stationary turbulence, we briefly give an interpretation of (2.13) for the case of freely decaying turbulence. The last term within the parentheses is the total amount of potential energy which has been dissipated up to time $t$. Clearly, this term goes to a constant as $t \rightarrow \infty$ and this constant must be some fraction, $a < 1$, of the total initial energy $E_K(0) + E_P(0)$. The two middle terms both go to zero as $t \rightarrow \infty$. Thus,

$$\langle\delta z^2\rangle \rightarrow 2\left[E_P(0) + a(E_K(0) + E_P(0))\right]/N^2 \tag{2.14}$$

as $t \rightarrow \infty$. A similar result was also obtained by Kaneda & Ishida (2000), using rapid distortion analysis combined with a statistical model. Their analysis suggests that the fraction $a$ is not universal, but varies with the initial conditions. Indeed, direct numerical simulations of decaying stratified turbulence (Kimura & Herring 1996; Kaneda & Ishida 2000; Venayagamoorthy & Stretch 2006) show that the mean-square vertical displacement of fluid particles takes a constant value at large times and that this value is proportional to $N^{-2}$. This was also demonstrated in an experiment on freely decaying stratified turbulence by Britter *et al.* (1983).

In the case where there is a continuous source of energy the turbulence can reach a stationary or quasi-stationary state where $\epsilon_P$ can be regarded as constant in time. In this case equation (2.13) can be written as

$$\langle \delta z^2 \rangle = \frac{\langle \delta b^2 \rangle}{N^2} + \frac{2}{N^2}\epsilon_P t\,, \tag{2.15}$$

where $\langle \delta b^2 \rangle = \langle (b(t) - b(0))^2 \rangle$ is the Lagrangian buoyancy structure function. It is obvious that the first term in (2.15) is identical to the term obtained by integrating (2.2) without including the molecular diffusion term, that is in case of adiabatic motion. In the case of a completely stationary process we have

$$\lim_{t/T\to\infty} \frac{\langle \delta b^2 \rangle}{N^2} = \frac{4E_P}{N^2}\,. \tag{2.16}$$

It is therefore unlikely that this term can give rise to any net gradient diffusion. We interpret this term as the adiabatic contribution to the mean-square displacement. The second term, on the other hand, is obtained as a result of including molecular diffusion in equation (2.2). It grows linearly with $t$, at least from $t \geqslant N^{-1}$ and as long as $\epsilon_P$ can be regarded as constant. We interpret this term as the diabatic contribution to the mean-square displacement, giving rise to irreversible mixing. We write it as

$$\langle \delta z^2 \rangle_{mix} = \frac{2}{N^2}\epsilon_P t\,. \tag{2.17}$$

Substituting this expression into (1.2), we obtain the Osborn & Cox (1972) expression for the eddy diffusivity. Note that the Osborn & Cox diffusivity is obtained here by considering the physics of fluid particle dispersion at time and length scales which are characteristic for the random fluid particle movements that mix the fluid, rather than assuming a global energy balance over much larger time and length scales.

### 2.2. *The adiabatic contribution*

In some cases the adiabatic mean-square displacement is mainly caused by linear gravity waves, as suggested by Lien & d'Asaro (2004). Linear gravity wave motion is adiabatic, oscillatory and cannot therefore cause any net diffusion. In other cases, however, the dynamics of adiabatic displacements are dominated by stratified turbulence (Riley & deBruynKops 2003; Lindborg 2006).

Somewhat popularized, stratified turbulence consists of 'pancake eddies' whose horizontal length scales are between the Ozmidov length scale, $l_O = \epsilon_K^{1/2}/N^{3/2}$, and the length scale $l_h = E_K^{3/2}/\epsilon_K$ and whose vertical scales are between the Ozmidov length scale and the scale $l_v = E_K^{1/2}/N$. The ratio $l_v/l_h$ defines a Froude number

$$F_h = \frac{\epsilon_K}{N E_K}\,, \tag{2.18}$$

which is much smaller than unity. In the upper troposphere and the lower stratosphere (see Lindborg 2006) $l_h$ is of the order of 500 km, $l_v$ is of the order of 1 or a few km, $l_O$ is of the order of 1 to 10 m and $F_h$ of the order of $10^{-3}$. The eddies of stratified turbulence are undergoing a forward energy cascade which is produced by strong nonlinear interactions, just as in three-dimensional Kolmogorov turbulence. The horizontal kinetic and potential energy spectra are of similar form as in three-dimensional turbulence, that is

$$E_K(k_h) = C_K \epsilon_K^{2/3} k_h^{-5/3}\,, \tag{2.19}$$

$$E_P(k_h) = C_P \frac{\epsilon_P}{\epsilon_K^{1/3}} k_h^{-5/3}, \tag{2.20}$$

where $C_K$ and $C_P$ are constants. In numerical simulations (Lindborg 2006; Lindborg & Brethouwer 2007) these two constants take the same value, $C_K \approx C_P \approx 0.5$. For the purpose of the present analysis we now make some assumptions about Lagrangian spectra of stratified turbulence. The Lagrangian frequency spectrum of a quantity is obtain by measuring the time series of that quantity following a fluid particle and then calculating the power spectrum of the signal. For Kolmogorov turbulence the Lagrangian frequency spectrum of velocity fluctuation has the form $\epsilon_K \omega^{-2}$, (see Monin & Yaglom 1975). This theoretical result has been confirmed for the spectrum of vertical velocity fluctuations in the upper boundary layer of the ocean (Lien, D'Asaro & Dairiki 1998). In close analogy with the theory of Kolmogorov turbulence we assume that there is an inertial range of frequencies where the Lagrangian spectra of velocity and buoyancy fluctuations have the form

$$E_{KL}(\omega) = C_{KL} \epsilon_K \omega^{-2}, \tag{2.21}$$

$$E_{PL}(\omega) = C_{PL} \epsilon_P \omega^{-2}, \tag{2.22}$$

where $C_{KL}$ and $C_{PL}$ are constants. We have included a factor of 1/2 in the definition, so that integration over all frequencies will give the total kinetic and potential energy. The form of the energy frequency spectra (2.21) and (2.22) can be motivated in a way similar to the corresponding kinetic energy spectrum of Kolmogorov turbulence. The turbulence is undergoing an energy cascade in which the transfer of kinetic and potential energy is equal to $\epsilon_K$ and $\epsilon_P$, respectively. From dimensional considerations it can be argued that the frequency spectra should have the form of (2.21) and (2.22) in the inertial range. In theory, the inertial range of stratified turbulence is within frequencies for which $T^{-1} \ll \omega \ll N$. In practice, however, the lower cutoff frequency can be supposed to be some fixed multiple of $T^{-1}$ and the higher cutoff frequency some fixed fraction of $N$.

D'Asaro & Lien (2000) show Lagrangian frequency spectra of horizontal kinetic energy measured in the ocean (see their figure 11), having the form $\omega^{-2}$ for three decades of frequencies, with two decades at lower frequencies than $N$ and one at higher frequencies. They interpret the lower-frequency part as an internal wave spectrum and the higher-frequency part as a spectrum of isotropic Kolmogorov turbulence. However, they also point out that 'the classification of a given flow as 'waves' or 'turbulence' is always problematic'. Typical fluid velocities in the observed field were found to be comparable to the internal wave phase speeds, pointing to the importance of strong nonlinearities. Thus, it does not seem unreasonable to interpret the lower-frequency part of the observed spectrum as a spectrum of stratified turbulence in accordance with (2.21). According to both interpretations there should be approximate equipartition between kinetic and potential energy for frequencies smaller than $N$. Therefore, the potential energy spectrum should have magnitude and shape similar to the kinetic energy spectrum according to both hypotheses. The difference is that the kinetic and potential energy spectra scale with $\epsilon_K$ and $\epsilon_P$ respectively, according to the stratified turbulence hypothesis, while no such scaling is predicted by the internal wave hypothesis.

Strictly speaking, the kinetic energy spectrum (2.21) is only the spectrum of horizontal velocity fluctuations. As we will argue in the next section, the spectrum of vertical velocity fluctuations of stratified turbulence will be flat or 'white-noise' in the inertial range, due to the strong damping of vertical motions. Therefore, the inertial-range dynamics of stratified turbulence is highly anisotropic, in contrast to the inertial-range dynamics of Kolmogorov turbulence, as also pointed out by Riley & Lindborg (2008).

In this analysis we will, in fact, only use the assumption (2.22). This assumption can also be formulated as (see Monin & Yaglom 1975, p. 90)

$$\langle \delta b^2 \rangle = C'_{PL} \epsilon_P t, \tag{2.23}$$

where $C'_{PL}$ is a constant which is related to $C_{PL}$ as

$$C'_{PL} = 2\pi C_{PL}. \tag{2.24}$$

Relation (2.23) is supposed to hold in an inertial range of time separations lying in between $N^{-1}$ and $T$.

Substituting the expression (2.23) into (2.15) we obtain

$$\langle \delta z^2 \rangle = (\pi C_{PL} + 1) 2 \epsilon_P t / N^2, \tag{2.25}$$

in this range of time scales. Here, the first term represents the adiabatic contribution and the second term represents the diabatic contribution to the mean-square particle displacement. The ratio between the two contributions is equal to $\pi C_{PL}$ and is universal, if the stratified turbulence assumption is true. Based on measurements of Lagrangian frequency spectra of vertical velocity fluctuations in the upper boundary layer of the ocean, Lien & D'Asaro (2004) found that the ratio between the measured mean-square displacement and that obtained from (2.17) was generally equal to about ten. Assuming that the adiabatic displacements in these measurements were dominated by stratified turbulence, we obtain $C_{PL} \approx 3$.

## 3. Comparison with previous analyses

In this paper, we have derived an analytical relation (2.13) for the evolution of $\langle \delta z^2 \rangle$, by integrating the dynamical equation for buoyancy fluctuations, neglecting two terms on the basis of order of magnitude estimates. The crucial assumption we make is that the interaction time scale for exchange of density between fluid elements is equal to the Kolmogorov time scale. If this assumption is correct, the derived relation has general validity.

In the case of freely decaying stratified turbulence we predict that for large times $\langle \delta z^2 \rangle$ should level off to a constant value determined by (2.14). This result is fully consistent with the prediction of Kaneda & Ishida (2000). The difference between our analysis and theirs is that we have directly integrated the equation for buoyancy fluctuations, while Kaneda & Ishida used rapid distortion analysis combined with statistical modelling. Our result can thus be viewed as analytical support for their model result.

Using a Langevin statistical model developed by Csanady (1964), Pearson *et al.* (1983) (henceforth referred to as PPH) derived a relation for the evolution of $\langle \delta z^2 \rangle$ for the case of stationary turbulence. For $t \gg N^{-1}$ their relation has the form

$$\langle \delta z^2 \rangle \sim \frac{\langle w^2 \rangle}{N^2} + \gamma^2 \frac{\langle w^2 \rangle}{N} t, \tag{3.1}$$

where $w$ is the vertical velocity fluctuation and $\gamma$ is a model parameter. The relation (3.1) has a form similar to our relation (2.15) for large times, since $\langle \delta z^2 \rangle$ is written as the sum of a constant term and a term growing linearly with time. However, there are some important differences. According to the analysis of PPH, $\langle \delta z^2 \rangle$ should reach the asymptotic form (3.1) in a time proportional to $N^{-1}$. In general, different curves of $\langle \delta z^2 \rangle$ should collapse if the time axis is scaled by $N^{-1}$. According to our analysis, the last term in (2.15) becomes equal to the second term at a time $t \approx 2E_P/\epsilon_P$. The asymptotic linear growth of $\langle \delta z^2 \rangle$ is therefore reached in a time proportional to

$$T \sim \frac{E_P}{\epsilon_P} \sim \frac{E_K}{\epsilon_K}. \tag{3.2}$$

This time scale is independent of $N$ and larger than $N$ by a factor $F_h^{-1}$. Thus, in the limit of strong stratification, our analysis suggests that $\langle \delta z^2 \rangle$ reaches its asymptotic form considerably later than suggested by the analysis of PPH. Moreover, a change of $N$, keeping all other parameters constant, would only affect the magnitude of the mean-square displacement but not the time scale of its evolution. Indeed, this is a very important difference between our result and the result of PPH. On the basis of existing experimental and numerical evidence it is not possible to decide which of the two predictions are correct. PPH reported some evidence in favour of their analysis, while Kimura & Herring (1996) noted that the damping rate of $\langle \delta z^2 \rangle$ was independent of $N$ in their direct numerical simulations, contrary to the prediction of PPH.

Another difference is that the first term in our relation (2.15) goes to the constant value $4E_P/N^2$ instead of $\langle w^2 \rangle / N^2$ as predicted by equation (3.1). For moderate stratification these two terms are presumably of the same order of magnitude. However, in the limit of strong stratification our term should be much larger than the corresponding term of PPH, since the ratio $E_P/E_K$ stays finite in the limit of small $F_h$ while the ratio $\langle w^2 \rangle / E_K$ goes to zero in the same limit (Brethouwer *et al.* 2007).

As for the last term in (2.15), which is growing linearly with $t$, we shall now show that it follows from our assumptions that this term can be rewritten into the same form as the corresponding term in the relation derived by PPH.

Taylor (1921) derived a formula for the growth of the mean square of fluid particle displacements, including the Lagrangian two-time velocity correlation function. Batchelor (1949) rewrote Taylor's formula in terms of the Lagrangian power frequency spectrum of velocity fluctations. According to this formula we can write

$$\langle \delta z^2 \rangle (t) = 4 \int_0^\infty E_{wL}(\omega) \frac{\sin^2(\omega t/2)}{\omega^2} \, \mathrm{d}\omega, \tag{3.3}$$

where $E_{wL}(\omega)$ is the power spectrum of vertical velocity fluctuations. Using this formula it is possible to show (Monin & Yaglom 1975, p. 528) that if $\langle \delta z^2 \rangle = K_1 t$ in a range of times $t$, then there is a corresponding range of frequencies where $E_{wL} = K_2$, with $K_2 = K_1/\pi$. From this relation we can deduce the general shape of $E_{wL}$. For $\omega \ll T^{-1}$ we should have

$$E_{wL}(\omega) = \frac{2\epsilon_P}{\pi N^2}, \tag{3.4}$$

For $T^{-1} \ll \omega \ll N$ we should have

$$E_{wL}(\omega) = \frac{2\epsilon_P(1 + \pi C_P)}{\pi N^2}. \tag{3.5}$$

For $N \ll \omega \ll \tau^{-1}$ we should have the frequency spectrum of Kolmogorov turbulence

$$E_{wL}(\omega) = \beta \epsilon_K \omega^{-2}, \tag{3.6}$$

where $\beta$ is the Lagrangian Kolmogorov constant. We can now estimate the total vertical velocity variance by direct integration of $E_{wL}$. The low frequency range will

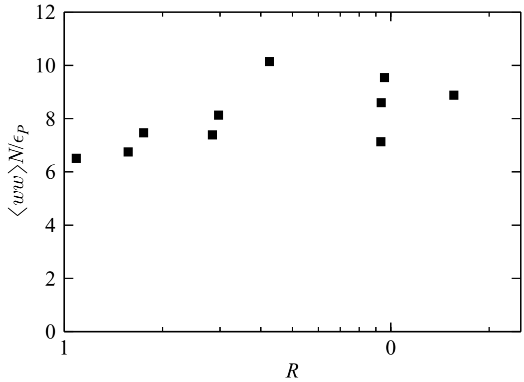


FIGURE 1. Vertical velocity variance normalized by $\epsilon_P/N$, as a function of buoyancy Reynolds number $R=\epsilon_K/(\nu N^2)$. From direct numerical simulations of strongly stratified stationary turbulence by Brethouwer *et al.* (2007).

give a negligible contribution. The contribution from the middle frequency range is estimated by integrating (3.5) from zero frequency to $N^{-1}$ and the contribution from the high frequency range is estimated by integrating (3.6) from $N^{-1}$ to infinity. In this way we find

$$\langle ww \rangle \approx \alpha \frac{\epsilon_P}{N}, \tag{3.7}$$

where

$$\alpha = 2C_P + \frac{2}{\pi} + \frac{\beta}{c} \approx 10. \tag{3.8}$$

Here, we have used the values $C_P = 3$, as estimated in the previous section, $\beta = 1.2$, as estimated by Tennekes & Lumley (1972)† and $c = 0.4$, as found in direct numerical simulations of Riley & deBruynKops (2003) and Brethouwer *et al.* (2007). If we instead use the standard value $c = 0.2$ for the mixing efficiency we obtain $\alpha \approx 13$, which in this context makes a small difference.

In figure 1, we have plotted $\langle w^2 \rangle N/\epsilon_P$ for different values of the buoyancy Reynolds number $R = \epsilon_K/(\nu N^2)$. The data points are calculated from direct numerical simulations of strongly stratified stationary turbulence previously reported by Brethouwer *et al.* (2007). The stratification is strong in all simulations and the value of $N$ is varied by a factor of 80 between the different simulations. In the simulations with highest buoyancy Reynolds number there is a narrow inertial range of stratified turbulence at length scales larger than the Ozmidov length scale. However, there is no classical Kolmogorov inertial range at smaller length scales. So it may be argued that the plot is not very relevant when we analyse the high-Reynolds-number limit. Nevertheless, we think that the weak variation of $\langle w^2 \rangle N/\epsilon_P$ with $R$ and the fact that the results are consistent with the estimated value $\alpha \approx 10$, lend some support to our analysis.

Substituting (3.7) into (2.15) we find

$$\langle \delta z^2 \rangle = \frac{\langle \delta b^2 \rangle}{N^2} + 2\alpha^{-1} \frac{\langle w^2 \rangle}{N} t. \tag{3.9}$$

† We have multiplied the value given by Tennekes & Lumley, $\beta = 1.8$, by 2/3, since we have used another definition of $\beta$.

The last term in this expression is equal to the last term in expression (3.1) taken from PPH, if we put $2\alpha^{-1} = \gamma^2$. With $\alpha = 10$, we obtain $\gamma = 0.45$, which is close to one of the values, $\gamma = 0.4$, estimated by PPH from observational evidence, but not very close to the other value, $\gamma = 0.1$.

## 4. Summary and conclusion

Apart from providing analytical support for the model result of Kaneda & Ishida (2000) for freely decaying stratified turbulence, we have made two well-defined predictions for stationary stratified turbulence. These predictions may be tested against observations or numerical simulations.

First, using the classical assumptions of local homogeneity and local isotropy, we have derived an expression (2.15) for the growth of mean-square displacements of fluid particles in stratified turbulence. The expression consists of two terms, which may be interpreted as the adiabatic and the diabatic dispersion. The diabatic term grows linearly with time, just like the mean quadratic displacements of molecular motions. This term determines the asymptotic linear growth of $\langle \delta z^2 \rangle$ and has the exact form $2\epsilon_P t/N^2$, including no unknown constant. Assuming that this term represents irreversible mixing, the Osborn & Cox (1972) expression for the eddy diffusivity is retained, by using Einstein's (1905) formula. We regard this as the most important new result. The adiabatic term goes to the finite limit (2.16) at time scales of the order of days and after this initial period the total growth is determined only by the diabatic term.

Second, based on a Lagrangian extension of the hypothesis of stratified turbulence at time scales between $N^{-1}$ and $T$, we have suggested that the adiabatic term also grows linearly with time and that the ratio between the adiabatic and the diabatic growth is universal for stratified turbulence in this range of time scales.

The reported observations by Lien & D'Asaro (2004) of vertical fluid particle dispersion in the ocean are fully consistent with the analysis presented here. We hope that the present paper will stimulate more experimental work of that kind.

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## Appendix

In this appendix we argue that the first three terms in (2.7) do not cancel each other in such a way that the fourth term becomes a leading-order term for $t \gg \tau$. The first two terms within the square brackets in (2.7) can be rewritten as

$$\frac{\mathrm{d}E_P}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\langle b(0)b(t)\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \delta b^2 \rangle, \tag{4.1}$$

where $\langle \delta b^2 \rangle = \langle (b(t) - b(0))^2 \rangle$ is the Lagrangian buoyancy structure function. This function is equal to zero for $t = 0$ and otherwise positive definite. It varies over the integral time scale of the turbulence, which we have assumed is equal to $T$ and should therefore be a monotonically increasing function for $t < T$. This means that (4.1) is positive and there will be no cancellation between the first three terms in (2.7) for $t < T$. In the limit $t \to \infty$, $\langle \delta b^2 \rangle$ goes to a constant in the case of stationary turbulence, so that (4.1) goes to zero and there will be no cancellation in this case either. For freely decaying turbulence, there may be a cancellation in the limit $t \to \infty$. This can be seen from the fact that the second term in (2.7) goes to zero in the limit $t \to \infty$, while the first term is negative. If the energy transfer from kinetic to potential energy goes to zero in the limit $t \to \infty$, the first two terms in (2.7) will, in fact, cancel the third term as $t \to \infty$. However, this possible cancellation will take place first in the limit $t \gg T$, while we have assumed that $F_1 \to 0$ in the limit $t \gg \tau$, which is reached much earlier. Therefore, the last term in (2.7) will be much smaller than the sum of the first three terms when $t \gg \tau$, in the case of both stationary and decaying turbulence.

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